Many-sorted Bound Founded Logic of Here-and-There: Work in Progress

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Abstract. The generate-and-test modeling methodology of Answer Set Programming (ASP) greatly benefits from the possibility to distinguish between founded and unfounded propositional variables (via choice rules). With the emergence of hybrid ASP reasoners, featuring numeric variables, the obvious question arises whether similar distinct treatments exist and whether they can be integrated in the same framework. The first question can already be affirmed since different hybrid solvers use different semantic approaches for dealing with numeric variables. In this paper, we start to address the second problem and present a many-sorted extension of the Bound Founded Logic of Here-and-There (HT_b). Follow-up work will detail how the distinct semantic approaches can be embedded into HT_b.

1 Introduction

The Logic of Here-and-There (HT; [7]) and its nonmonotonic extension, Equilibrium Logic [9], provide the logical foundations for Answer Set Programming (ASP; [8]). Specifically, the stable models of a logic program correspond to the equilibrium models of the corresponding set of implications. However, the scope of HT extends well beyond this traditional setting. It has also been extended to characterize extensions of ASP that incorporate foreign language constructs and corresponding inferences, such as temporal modalities and linear constraints [1, 4].

In what follows, we focus on the logics HT_b [3] and HT_c [6],[4] which provide a general framework for extending HT with constraints. Notably, HT_c has been used to give semantics to extensions of *clingo* such as *clingcon*, *flingo*, and *clingo*[DL]. All three systems can express rules like:

$$a \vee \neg a \qquad a \rightarrow x > 7$$

The semantics of *clingcon* produces infinitely many solutions: one type where a is true and x takes in turn all values greater than 7, and another where a is false

[4] HT_c stands for the Logic of Here-and-There with Constraints.

and x takes all possible integer values. In contrast, *flingo*'s semantics — which corresponds to basic HT_c — yields a single solution that makes a false and leaves x undefined while the solutions making a true are the same as obtained with *clingcon*. Finally, with HT_b, we obtain only two solutions: one where a is true and x assigned 8 and another where a is false and x is undefined.

The differences among these three systems can be explained by their varying degrees of foundedness. In ASP, all true atoms must be founded, meaning they are derivable via rules from facts. A similar principle applies to HT_c and HT_b: a variable like x takes an integer value only if its containing constraint atom $x > 7$ is derivable. If the constraint atom is unfounded, x remains undefined. In contrast, *clingcon* disconnects constrained atoms from foundedness, making the second rule equivalent to $a \wedge \neg x > 7 \rightarrow \bot$. HT_b strengthens the concept of foundedness in HT_c by extending it to include the ordering of integers. Specifically, if $x > 7$ is derivable, HT_c sanctions all values greater than 7, whereas HT_b permits only the smallest admissible value, 8.

The benefit of HT_b's approach can be seen when regarding difference constraints, like $x - y \leq 5$, as used in *clingo*[DL]. Such a constraint is satisfied by all pairs of integers with a difference of 5. Rather than considering this infinite set, one often applies a so-called "asap" approach, providing only the "smallest" among these pairs (cf. [4]).

Hence, when it comes to modeling, all three approaches are valuable, as we see in ASP where we can use founded and unfounded variables via choice rules. To provide this flexibility within a uniform framework, we introduce a many-sorted extension of HT_b (and HT_c) that enables the definition of variables over different ordered domains. Also, we show that HT_c can be embedded in this framework and particular variables treated accordingly.

2 Many-sorted, Bound Founded Logic of Here-and-There

We present below a many-sorted extension of HT_b, or in full detail, the Bound Founded Logic of Here-and-There, and elaborate on its properties.

We define the many-sorted language of HT_b[5] over a signature

$$\Sigma = \langle \mathcal{S}, (\mathcal{X}_s)_{s\in\mathcal{S}}, (\mathcal{D}_s, \preceq_s)_{s\in\mathcal{S}}, \mathcal{C} \rangle\,, \quad \text{where}$$

1. $\mathcal{S}$ is a set of sorts,
2. $(\mathcal{X}_s)_{s\in\mathcal{S}}$ is a partition of a set of variables,
3. $(\mathcal{D}_s, \preceq_s)_{s\in\mathcal{S}}$ is a collection of (partially) ordered non-empty domains, and
4. $\mathcal{C}$ is a set of constraint atoms over $\bigcup_{s\in\mathcal{S}} \mathcal{X}_s$ and $\bigcup_{s\in\mathcal{S}} \mathcal{D}_s$.

That is, each $\preceq_s$ is reflexive, anti-symmetric and transitive. In what follows, we often use Σ only and leave its constituents implicit.

For convenience, we let $\mathcal{X}$ and $\mathcal{D}$ stand for $\bigcup_{s\in\mathcal{S}} \mathcal{X}_s$ and $\bigcup_{s\in\mathcal{S}} \mathcal{D}_s$, respectively. Also, for simplicity, we associate each element of $\mathcal{D}$ with its representing constant.

[5] For simplicity, we refrain from using a new acronym for this many-sorted variant of HT_b.

The specific syntax of constraint atoms in $\mathcal{C}$ is left open but is assumed to refer to elements of $\mathcal{X}$ and $\mathcal{D}$. Thus, an atom can be understood to hold or not once all variables in it are substituted by domain elements.

We define a (partial) *valuation* over Σ as a relation $v \subseteq \mathcal{X} \times \mathcal{D}$, such that

1. if $(x, d) \in v$ and $(x, d') \in v$, then $d = d'$ for all $x \in \mathcal{X}$, $d, d' \in \mathcal{D}$, and
2. if $(x, d) \in v$, then $(x, d) \in \mathcal{X}_s \times \mathcal{D}_s$ for some $s \in \mathcal{S}$.

The first condition makes sure that v behaves functionally, while the second ensures that it respects sort information. Since a valuation v behaves functionally, we can also write $v(x) = d$ if $(x, d) \in v$ and $v(x) = \boldsymbol{u}$ otherwise, where $\boldsymbol{u}$ stands for the special domain element *undefined*. We let $\mathcal{V}_\Sigma$ stand for the set of valuations over Σ but we drop the subscript and just write $\mathcal{V}$, whenever clear from context.

We define the *downward closure* of a valuation v over signature Σ as

$$v{\downarrow_\Sigma} = \{(x, d) \mid (x, c) \in v, x \in \mathcal{X}_s, d \in \mathcal{D}_s, d \preceq_s c\} .$$

The purpose of this closure is to enable the comparison of valuations in terms of set inclusion. For instance, the downward closure of one valuation is strictly contained in that of another, only if it assigns strictly smaller values to all variables. When clear from context, we write v instead of $v{\downarrow_\Sigma}$.

A formula over signature Σ is defined as

$$\varphi ::= \bot \mid c \in \mathcal{C} \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi .$$

As usual, we define $\top$ as $\bot \rightarrow \bot$ and $\neg\varphi$ as $\varphi \rightarrow \bot$ for any formula φ. A theory is a set of formulas.

Satisfaction of constraint atoms in $\mathcal{C}$ is defined wrt *denotations* over Σ which are functions $[\![\cdot]\!]_\Sigma : \mathcal{C} \rightarrow 2^{\mathcal{V}_\Sigma}$ mapping atoms to sets of valuations. Again, we drop the subscript and just write $[\![\cdot]\!]$ whenever clear from context.

We define an interpretation over Σ as a pair $\langle h, t\rangle$ of valuations over Σ such that $h{\downarrow_\Sigma} \subseteq t{\downarrow_\Sigma}$.

Now that all key concepts have been enriched with sorts, the satisfaction of formulas in HT_b is defined as before as follows.

Definition 1. *Let $\langle h, t\rangle$ be an interpretation over Σ and φ be a formula. Then, $\langle h, t\rangle$ satisfies φ, written $\langle h, t\rangle \models \varphi$, if the following holds:*

1. $\langle h, t\rangle \not\models \bot$
2. $\langle h, t\rangle \models c$ *iff* $v \in [\![c]\!]_\Sigma$ *for atom* $c \in \mathcal{C}$ *and for all* $v \in \{h, t\}$
3. $\langle h, t\rangle \models \varphi_1 \wedge \varphi_2$ *iff* $\langle h, t\rangle \models \varphi_1$ *and* $\langle h, t\rangle \models \varphi_2$
4. $\langle h, t\rangle \models \varphi_1 \vee \varphi_2$ *iff* $\langle h, t\rangle \models \varphi_1$ *or* $\langle h, t\rangle \models \varphi_2$
5. $\langle h, t\rangle \models \varphi_1 \rightarrow \varphi_2$ *iff* $\langle v, t\rangle \not\models \varphi_1$ *or* $\langle v, t\rangle \models \varphi_2$ *for both* $v \in \{h, t\}$

We call $\langle h, t\rangle$ a model of a theory Γ, if $\langle h, t\rangle \models \varphi$ for all φ in Γ.

Definition 2. *An interpretation $\langle t, t\rangle$ over Σ is an* equilibrium model *of a theory Γ over Σ, if $\langle t, t\rangle \models \Gamma$ and there is no valuation h over Σ such that $h{\downarrow_\Sigma} \subset t{\downarrow_\Sigma}$ and $\langle h, t\rangle \models \Gamma$.*

If $\langle t, t\rangle$ is an equilibrium model of Γ over Σ, then we call t a *Σ-stable model* of Γ.

Let us illustrate how equilibrium models depend on the chosen order in HT_b.

Example 1. Consider the theory $\Gamma = \{x \geq 1\}$ over the single-sorted signature

$$\Sigma_1 = \langle \{s_1\}, \{x\}, (\{1,2\}, \{(1,1),(2,2)\}), \{x \geq 1\}\rangle \,.$$

The Σ_1-stable models of Γ are $t_1 = \{(x,1)\}$ and $t_2 = \{(x,2)\}$.

Their downward closures are $t_i{\downarrow_{\Sigma_1}} = t_i$ for $i \in \{1,2\}$. Thus, in both cases the only possible valuation with a smaller downward closure is $\emptyset$. Given that $\langle \emptyset, t_i\rangle \not\models \Gamma$ for $i \in \{1,2\}$, we get that both t_1 and t_2 are stable models.

However, when considering the signature

$$\Sigma_2 = \langle \{s_2\}, \{x\}, (\{1,2\}, \leq), \{x \geq 1\}\rangle,$$

obtained from Σ_1 by using the "lesser or equal" relation, the second stable model t_2 disappears and we only obtain t_1 as Σ_2-stable model. This is because we now get that

$$\{(x,1)\} = t_1{\downarrow_{\Sigma_2}} \subset t_2{\downarrow_{\Sigma_2}} = \{(x,1),(x,2)\}$$

and $\langle t_1, t_2\rangle \models \Gamma$.

Note that it is possible to obtain both stable models in Σ_2 by adding the formula $(x \leq 2) \vee \neg(x \leq 2)$ to Γ.

The example illustrates that a stronger order, having more comparable domain elements, induces a stronger minimization and thus yields fewer stable models. Conversely, stable models are preserved when reducing the order. The following result makes this precise.

Proposition 1. *Let $\Sigma_i = \langle \mathcal{S}, (\mathcal{X}_s)_{s\in\mathcal{S}}, (\mathcal{D}_s, \preceq_s{}^i)_{s\in\mathcal{S}}, \mathcal{C}\rangle$ for $i \in \{1,2\}$ be two signatures such that $\preceq_s{}^1 \subseteq \preceq_s{}^2$ for all $s \in \mathcal{S}$, and let Γ be a theory over Σ_1 (or Σ_2).*

If t is a Σ_2-stable model of Γ, then t is a Σ_1-stable model of Γ.

Next, we show that the many-sorted semantics of HT_b can be reduced to a single-sorted one whenever all pairs of subdomains and their associated orders behave the same over any common domain elements. We now make this precise.

Define the projection of an order $\preceq_s$ onto a set V as

$$\preceq_s|_V = \{(d,d') \mid (d,d') \in \preceq_s \text{ and } d,d' \in V\}\,.$$

Then, we say that two domain-order pairs $(\mathcal{D}_1, \preceq_1)$ and $(\mathcal{D}_2, \preceq_2)$ are *compatible* if they satisfy the following condition

$$\preceq_1|_{\mathcal{D}_1\cap\mathcal{D}_2} = \preceq_2|_{\mathcal{D}_1\cap\mathcal{D}_2}\,. \tag{1}$$

This ensures that the orders have the same behavior over the common domain elements, as well as that the union of the two orders is reflexive and anti-symmetric. We say that two sorts s_1 and s_2 are compatible, if their corresponding domain-order pairs $(\mathcal{D}_{s_1}, \preceq_{s_1})$ and $(\mathcal{D}_{s_2}, \preceq_{s_2})$ are compatible.

To illustrate why (1) is necessary, consider the two domain-order pairs $(\mathbb{N}, \leq_{\mathbb{N}})$ and $(\mathbb{Z}, \leq_{\mathbb{Z}})$ which are compatible as their intersection is $\mathbb{N}$ and both orders coincide over it. On the contrary, the pairs $(\mathbb{N}, \leq_{\mathbb{N}})$ and $(\mathbb{N}, \{(n,n) \mid n \in \mathbb{N}\})$ are not compatible since $\leq_{\mathbb{N}}$ and $\{(n,n) \mid n \in \mathbb{N}\}$ do not coincide over $\mathbb{N}$. Note that (1) does not ensure transitivity of the new order, if we simply take the union. Consider two domain-order pairs $(\mathcal{D}_1, \preceq_1) = (\{1,2\}, \{(1,1),(1,2),(2,2)\})$ and $(\mathcal{D}_2, \preceq_2) = (\{2,3\}, \{(2,2),(2,3),(3,3)\})$. They are compatible because $\mathcal{D}_1 \cap \mathcal{D}_2 = \{2\}$ and the projections of the orders onto the intersection are $\preceq_1|_{\mathcal{D}_1 \cap \mathcal{D}_2} = \{(2,2)\} = \preceq_2|_{\mathcal{D}_1 \cap \mathcal{D}_2}$. However, $\preceq_1 \cup \preceq_2$ is not transitive. Therefore, we take the transitive closure $(\preceq_1 \cup \preceq_2)^+$ as the new, common order for both sorts.

Lemma 1. *Let $\Sigma = \langle \mathcal{S}, (\mathcal{X}_s)_{s\in\mathcal{S}}, (\mathcal{D}_s, \preceq_s)_{s\in\mathcal{S}}, \mathcal{C}\rangle$ be a signature such that the sorts in $\mathcal{S}$ are pairwise-compatible. Then, the order $\preceq_{s'} = (\bigcup_{s\in\mathcal{S}} \preceq_s)^+$, where $^+$ denotes the transitive closure, is reflexive, anti-symmetric, and transitive.*

Additionally, we need to introduce additional axioms to ensure that variables in the new single-sorted signature respect their domains from the original signature. For this we define a specific kind of constraint atom.

Given a signature Σ, a variable $x \in \mathcal{X}_s$ and a subset $\mathcal{D}'_s \subseteq \mathcal{D}_s$, we define the associated constraint atom $x : \mathcal{D}'_s$ with denotation

$$[\![\, x : \mathcal{D}'_s \,]\!]_\Sigma = \{v \in \mathcal{V}_\Sigma \mid (x, d') \in v, d' \in \mathcal{D}'_s\}\,.$$

Note that for each $v \in [\![\, x : \mathcal{D}'_s \,]\!]_\Sigma$ this implies that x is defined in v.

Proposition 2. *Let $\Sigma = \langle \mathcal{S}, (\mathcal{X}_s)_{s\in\mathcal{S}}, (\mathcal{D}_s, \preceq_s)_{s\in\mathcal{S}}, \mathcal{C}\rangle$ be a signature such that the sorts in $\mathcal{S}$ are pairwise-compatible, and let the corresponding single-sorted signature be $\Sigma' = \langle \{s'\}, \bigcup_{s\in\mathcal{S}} \mathcal{X}_s, (\mathcal{D}_{s'}, \preceq_{s'}), \mathcal{C}\rangle$, where $s' \notin \mathcal{S}$, $\mathcal{D}_{s'} = \bigcup_{s\in\mathcal{S}} \mathcal{D}_s$, and $\preceq_{s'} = (\bigcup_{s\in\mathcal{S}} \preceq_s)^+$, with $^+$ denoting the transitive closure.*

Let Γ be a theory over Σ and let $\Gamma' = \Gamma \cup \bigcup_{s\in\mathcal{S}}\{x : \mathcal{D}_{s'} \to x : \mathcal{D}_s \mid x \in \mathcal{X}_s\}$. Then, t is a Σ-stable model of Γ iff t is a Σ'-stable model of Γ'.

3 Embedding HT$_c$ into many-sorted HT$_b$

In this section, we show how HT$_c$ can be embedded into many-sorted HT$_b$.

In HT$_c$, a signature consists of a triple $\langle \mathcal{X}, \mathcal{D}, \mathcal{C}\rangle$, where $\mathcal{X}$ is a set of variables, $\mathcal{D}$ is a non-empty domain, and $\mathcal{C}$ is a set of constraint atoms. Valuations and denotations are defined as in HT$_b$ when confined to a single sort; similarly, the satisfaction relation in HT$_c$, written $\models_c$, is analogous to HT$_b$, For completeness, these definitions are reproduced in Section A.3.

The main difference of HT$_c$ to (many-sorted) HT$_b$ is that its domain is unordered (and there are no sorts). Accordingly, for defining interpretations and equilibrium models, valuations need merely be compared in terms of their degree of undefinedness, which can be accomplished by plain set inclusion. Hence, in HT$_c$, an interpretation over $\langle \mathcal{X}, \mathcal{D}, \mathcal{C}\rangle$ is a pair $\langle h, t\rangle$ of valuations over $\langle \mathcal{X}, \mathcal{D}, \mathcal{C}\rangle$ such that $h \subseteq t$. An interpretation $\langle t, t\rangle$ over $\langle \mathcal{X}, \mathcal{D}, \mathcal{C}\rangle$ is an equilibrium model of a theory Γ in HT$_c$ if $\langle t, t\rangle \models_c \Gamma$ and there is no $h \subset t$ such that $\langle h, t\rangle \models_c \Gamma$. As above, we call t a stable model of Γ in HT_c, if $\langle t, t\rangle$ is an equilibrium model of Γ in HT_c. As illustrated in Example 1 and made precise in Proposition 1, HT_c gives us more stable models as it employs a weaker minimization on interpretations.

We show below that any theory in HT_c can be expressed as a single-sorted theory in HT_b such that their equilibrium models coincide. For any signature $\langle \mathcal{X}, \mathcal{D}, \mathcal{C}\rangle$ in HT_c, we define a single-sorted signature $\Sigma_{\langle \mathcal{X}, \mathcal{D}, \mathcal{C}\rangle} = \langle \{s\}, \mathcal{X}_s, (\mathcal{D}_s, id_s), \mathcal{C}\rangle$ in HT_b with $\mathcal{X}_s = \mathcal{X}$, $\mathcal{D}_s = \mathcal{D}$, and $id_s = \{(d,d) \mid d \in \mathcal{D}_s\}$. Note that any formula in HT_c over $\langle \mathcal{X}, \mathcal{D}, \mathcal{C}\rangle$ is also a formula in HT_b over $\Sigma_{\langle \mathcal{X}, \mathcal{D}, \mathcal{C}\rangle}$, and vice versa, and thus we can consider the same theory Γ over both signatures.

Theorem 1. *Let $\langle \mathcal{X}, \mathcal{D}, \mathcal{C}\rangle$ be an HT_c-signature, and let the corresponding HT_b-signature be $\Sigma_{\langle \mathcal{X}, \mathcal{D}, \mathcal{C}\rangle} = \langle \{s\}, \mathcal{X}_s, (\mathcal{D}_s, id_s), \mathcal{C}\rangle$ with $\mathcal{X}_s = \mathcal{X}$ and $\mathcal{D}_s = \mathcal{D}$. Let Γ be a theory over $\langle \mathcal{X}, \mathcal{D}, \mathcal{C}\rangle$ (or $\Sigma_{\langle \mathcal{X}, \mathcal{D}, \mathcal{C}\rangle}$).*

Then, t is a stable model of Γ in HT_c iff t is a $\Sigma_{\langle \mathcal{X}, \mathcal{D}, \mathcal{C}\rangle}$-stable model of Γ in HT_b.

In view of Theorem 1, we can also define a fragment of many-sorted HT_b that corresponds to a many-sorted version of HT_c. The semantics of many-sorted HT_c is then captured in HT_b by the signature $\langle \mathcal{S}, (\mathcal{X})_{s\in\mathcal{S}}, (\mathcal{D}_s, id_s)_{s\in\mathcal{S}}, \mathcal{C}\rangle$. For such signatures, valuations are equal to their downward closures.

Proposition 3. *Let $\Sigma = \langle \mathcal{S}, (\mathcal{X})_{s\in\mathcal{S}}, (\mathcal{D}_s, id_s)_{s\in\mathcal{S}}, \mathcal{C}\rangle$ be an HT_b-signature and let $v \in \mathcal{V}_\Sigma$ be a valuation over Σ.*

Then, we get that $v{\downarrow} = v$.

4 Conclusion

We presented a many-sorted extension of the Bound Founded Logic of Here-and-There (HT_b) and demonstrated how the related logic of Here-and-There with Constraints (HT_c) can be embedded within it. This embedding provides insight into the formation of equilibrium models in both logics, depending on whether they rely upon ordered domains or not. This also reflects the principle of stable model formation in ASP, where atoms remain 'false' unless they are provably 'true', which is analogous to using an ordering on Boolean values where 'true' is greater than 'false'—an observation first made in [2]. Since HT_c has already been used to characterize specific hybrid ASP extensions and provide semantics for corresponding extensions of *clingo*, we are confident that our many-sorted approach provides a unified, integrative framework for hybrid ASP. Future work will involve applying our framework to a broader range of hybrid ASP extensions and further exploring its theoretical properties.

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A Proofs

For the remainder of the paper we provide the following auxiliary definitions:

For a valuation v (in HT_b or HT_c), we define $dom(v) = \{x \mid (x, d) \in v\}$ to be the set of variables occurring in a valuation v. For a formula φ (in HT_b or HT_c), we define $vars(\varphi) \subseteq \mathcal{X}$ to be the set of variables occurring in φ. We start with the following auxiliary Lemma

Lemma 2. *Let v, v' be two valuations over a signature Σ. Then, we get that $v = v'$ iff $v{\downarrow_\Sigma} = v'{\downarrow_\Sigma}$.*

Proof of Lemma 2 The *left-to-right* direction follows directly from the definition of the downward closure. Assume $v = v'$. Then, we get that

$$\begin{aligned} v{\downarrow_\Sigma} &= \{(x, d) \mid (x, c) \in v, x \in \mathcal{X}_s, d \in \mathcal{D}_s, d \preceq_s c\} \\ &= \{(x, d) \mid (x, c) \in v', x \in \mathcal{X}_s, d \in \mathcal{D}_s, d \preceq_s c\} \\ &= v'{\downarrow_\Sigma} \end{aligned}$$

For the *right-to-left* direction, assume $v{\downarrow_\Sigma} = v'{\downarrow_\Sigma}$. Without loss of generality, assume that there exists an $x \in \mathcal{X}_s$ for some sort $s \in \mathcal{S}$ such that $(x, c) \in v$ and

$(x, c) \notin v'$. Then, $(x, c) \in v{\downarrow_\Sigma} = v'{\downarrow_\Sigma}$, and by definition of the downward closure, there exists some $c' \in \mathcal{D}_s$ such that $(x, c') \in v'$ and $c \preceq_s c'$. From $(x, c') \in v'$ it follows that $(x, c') \in v'{\downarrow_\Sigma} = v{\downarrow_\Sigma}$, and thus by definition of the downward closure and since $(x, c) \in v$, we get that $c' \preceq_s c$. Then, we have that $c \preceq_s c'$ and $c' \preceq_s c$, and since $\preceq_s$ is anti-symmetric, it follows that $c = c'$. Therefore, $(x, c) \in v'$ which is a contradiction. We conclude that $v = v'$. □

Proof of Lemma 3 Let $v \in \mathcal{V}_\Sigma$ be a valuation over Σ.

Then,

$$\begin{aligned} v{\downarrow} &= \{(x, d) \mid (x, c) \in v, x \in \mathcal{X}_s, d \in \mathcal{D}_s, d\ id_s\ c\} \\ &= \{(x, c) \mid (x, c) \in v\} \\ &= v \end{aligned}$$

□

A.1 Proof of Proposition 1

For the remainder of the proof let $\Sigma_i = \langle \mathcal{S}, (\mathcal{X}_s)_{s\in\mathcal{S}}, (\mathcal{D}_s, \preceq_s^i)_{s\in\mathcal{S}}, \mathcal{C}\rangle$ for $i \in \{1, 2\}$ be two signatures such that $\preceq_s^1 \subseteq \preceq_s^2$ for all $s \in \mathcal{S}$.

Note that since Σ_1 and Σ_2 have the same sorts, variables, and domain elements, it follows that any valuation v over Σ_1 is also a valuation over Σ_2, and vice versa.

Lemma 3. *For any valuation v over Σ_1 (or Σ_2), it holds that $v{\downarrow_{\Sigma_1}} \subseteq v{\downarrow_{\Sigma_2}}$.*

Proof. Pick any $(x, d) \in v{\downarrow_{\Sigma_1}}$. Then, we get that $x \in \mathcal{X}_s$ and $d \in \mathcal{D}_s$ for some $s \in \mathcal{S}$. By definition of the downward closure we get that there exists some $(x, c) \in v \cap \mathcal{X}_s \times \mathcal{D}_s$ such that $d \preceq_s^1 c$. Since $\preceq_s^1 \subseteq \preceq_s^2$, we get that $d \preceq_s^2 c$. It follows that $(x, d) \in v{\downarrow_{\Sigma_2}}$.

Lemma 4. *For any two valuations v and v' over Σ_1 (or Σ_2), it holds that $v{\downarrow_{\Sigma_1}} \subseteq v'{\downarrow_{\Sigma_2}}$ implies $v{\downarrow_{\Sigma_2}} \subseteq v'{\downarrow_{\Sigma_2}}$.*

Proof. Assume $v{\downarrow_{\Sigma_1}} \subseteq v'{\downarrow_{\Sigma_2}}$. Pick any $(x, d) \in v{\downarrow_{\Sigma_2}}$. Then, we get that $x \in \mathcal{X}_s$ and $d \in \mathcal{D}_s$ for some $s \in \mathcal{S}$. By definition of the downward closure, there exists some $(x, c) \in v \cap \mathcal{X}_s \times \mathcal{D}_s$ such that $d \preceq_s^2 c$. Since $\preceq_s^1$ is reflexive and $c \in \mathcal{D}_s$, we get that $c \preceq_s^1 c$. From $(x, c) \in v$ and $c \preceq_s^1 c$, it follows that $(x, c) \in v{\downarrow_{\Sigma_1}}$. By assumption, it follows that $(x, c) \in v'{\downarrow_{\Sigma_2}}$. Then, by definition of the downward closure, we get that there exists some $(x, c') \in v' \cap \mathcal{X}_s \times \mathcal{D}_s$ such that $c \preceq_s^2 c'$. From $c \preceq_s^2 c'$ and $d \preceq_s^2 c$ and from the transitivity of $\preceq_s^2$, it follows that $d \preceq_s^2 c'$. Therefore, $(x, d) \in v'{\downarrow_{\Sigma_2}}$.

Corollary 1. *If $\langle h, t\rangle$ is an interpretation over Σ_1 it is also an interpretation over to Σ_2.*

Proof. Let $\langle h, t\rangle$ be an interpretation over Σ_1. By definition, $h{\downarrow_{\Sigma_1}} \subseteq t{\downarrow_{\Sigma_1}}$. By Lemma 3, it follows that $t{\downarrow_{\Sigma_1}} \subseteq t{\downarrow_{\Sigma_2}}$. It follows that $h{\downarrow_{\Sigma_1}} \subseteq t{\downarrow_{\Sigma_2}}$. By Lemma 4 and taking $v = h$ and $v' = t$, we get that $h{\downarrow_{\Sigma_2}} \subseteq t{\downarrow_{\Sigma_2}}$. Therefore, $\langle h, t\rangle$ is an interpretation over Σ_2.

Proof of Proposition 1 Let t be a Σ_2-stable model of Γ. Then, by Definition 2, we get that $\langle t, t\rangle \models \Gamma$ and there is no $h{\downarrow_{\Sigma_2}} \subset t{\downarrow_{\Sigma_2}}$ such that $\langle h, t\rangle \models \Gamma$. The only difference between the two signatures are the orders $\preceq_s^1$ and $\preceq_s^2$, which do not influence the satisfaction relation. Therefore, it is sufficient to show that there is no valuation h' over Σ_1 such that $h'{\downarrow_{\Sigma_1}} \subset t{\downarrow_{\Sigma_1}}$ and $\langle h', t\rangle \models \Gamma$.

Let h' be valuation over Σ_1 such that $h'{\downarrow_{\Sigma_1}} \subset t{\downarrow_{\Sigma_1}}$. By Lemma 2, we get that $h' \neq t$. From $\preceq_s^1 \subseteq \preceq_s^2$ and by Lemma 3, it follows that $t{\downarrow_{\Sigma_1}} \subseteq t{\downarrow_{\Sigma_2}}$ which implies that $h'{\downarrow_{\Sigma_1}} \subset t{\downarrow_{\Sigma_2}}$. Using this and Lemma 4, we get that $h'{\downarrow_{\Sigma_2}} \subseteq t{\downarrow_{\Sigma_2}}$. From $h' \neq t$ and Lemma 2, it follows that $h'{\downarrow_{\Sigma_2}} \subset t{\downarrow_{\Sigma_2}}$. Then, from our assumption that t is a Σ_2-stable model, we get that $\langle h', t\rangle \not\models \Gamma$. It follows that t is a Σ_1-stable model of Γ. □

A.2 Proof of Proposition 2

For the remainder of the proof let $\Sigma = \langle \mathcal{S}, (\mathcal{X}_s)_{s\in\mathcal{S}}, (\mathcal{D}_s, \preceq_s)_{s\in\mathcal{S}}, \mathcal{C}\rangle$ be a signature such that the sorts in $\mathcal{S}$ are pairwise-compatible. Let $\Sigma' = \langle \{s'\}, \bigcup_{s\in\mathcal{S}} \mathcal{X}_s, (\mathcal{D}_{s'}, \preceq_{s'}), \mathcal{C}\rangle$ be a signature with $s' \notin \mathcal{S}$, $\mathcal{D}_{s'} = \bigcup_{s\in\mathcal{S}} \mathcal{D}_s$, $\preceq_{s'} = (\bigcup_{s\in\mathcal{S}} \preceq_s)^+$, where $^+$ denotes the transitive closure.

Proof of Lemma 1

1. To prove reflexivity, pick any domain element $d \in \mathcal{D}_{s'}$. Then, by construction of Σ', it follows that there exists some $s \in \mathcal{S}$ such that $d \in \mathcal{D}_s$. Since $\preceq_s$ is reflexive, it follows that $d \preceq_s d$, and since $\preceq_{s'} = \bigcup_{s\in\mathcal{S}} \preceq_s$ we get that $d \preceq_{s'} d$
2. For anti-symmetry, pick any two $d, d' \in \mathcal{D}_{s'}$ and assume $d \preceq_{s'} d'$ and $d' \preceq_{s'} d$. Then, there exists $s_1, s_2 \in \mathcal{S}$ such that $d \preceq_{s_1} d'$ and $d' \preceq_{s_2} d$. Either we get that $s_1 = s_2$ which implies $d \preceq_{s_1} d'$ and $d' \preceq_{s_1} d$, or, if $s_1 \neq s_2$, it follows that $d, d' \in \mathcal{D}_{s_1} \cap \mathcal{D}_{s_2}$. Since s_1 and s_2 are compatible (first condition), we get that $\preceq_{s_1}|_{\mathcal{D}_{s_1}\cap\mathcal{D}_{s_2}} = \preceq_{s_2}|_{\mathcal{D}_{s_1}\cap\mathcal{D}_{s_2}}$ from which it follows that $d \preceq_{s_2} d'$ and $d' \preceq_{s_1} d$. In both cases, we get that $d \preceq_{s_1} d'$ and $d' \preceq_{s_1} d$ and since $\preceq_{s_1}$ is anti-symmetric, it follows that $d = d'$.
3. Transitivity follows directly because $\preceq_{s'}$ is defined as the transitive closure of $\bigcup_{s\in\mathcal{S}} \preceq_s$.

□

Lemma 5. *If v is a valuation over Σ, then v is a valuation over Σ' and $v{\downarrow_{\Sigma}} \subseteq v{\downarrow_{\Sigma'}}$.*

Proof. Let v be a valuation over Σ. Since $\mathcal{X}_{s'} = \bigcup_{s\in\mathcal{S}} \mathcal{X}_s$, and $\mathcal{D}_{s'} = \bigcup_{s\in\mathcal{S}} \mathcal{D}_s$, and there is only a single sort s' in Σ', it follows that v is also a valuation over Σ'.

Next, pick any $(x, d) \in v{\downarrow_\Sigma}$. Then, we get that $x \in \mathcal{X}_s$ and $d \in \mathcal{D}_s$ for some $s \in \mathcal{S}$. By definition of the downward closure, there exists some $(x, c) \in v \cap \mathcal{X}_s \times \mathcal{D}_s$ such that $d \preceq_s c$. From the construction of Σ' it follows that $(x, c) \in v \cap (\mathcal{X}_{s'} \times \mathcal{D}_{s'})$ and from $\preceq_{s'} = \bigcup_{s\in\mathcal{S}} \preceq_s$ it follows that $d \preceq_{s'} c$. Therefore, $(x, d) \in v{\downarrow_{\Sigma'}}$.

Lemma 6. *$\langle h, t\rangle$ is an interpretation over Σ iff $\langle h, t\rangle$ is an interpretation over Σ' which satisfies $\langle h, t\rangle \models \bigcup_{s\in\mathcal{S}} \{x : \mathcal{D}_{s'} \to x : \mathcal{D}_s \mid x \in \mathcal{X}_s\}$.*

Proof. For the *left-to-right* direction, assume that $\langle h, t\rangle$ is an interpretation over Σ. By definition of a valuation, we get that $\langle h, t\rangle \models \bigcup_{s\in\mathcal{S}} \{x : \mathcal{D}_{s'} \to x : \mathcal{D}_s \mid x \in \mathcal{X}_s\}$ because variables can only take on values from their sort and $\langle h, t\rangle$ is an interpretation over Σ. By Lemma 5 it follows that h and t are valuations over Σ'.

To show that $\langle h, t\rangle$ is an interpretation over Σ', we need to show that the valuations h and t satisfy $h{\downarrow_{\Sigma'}} \subseteq t{\downarrow_{\Sigma'}}$. Pick any $(x, d) \in h{\downarrow_{\Sigma'}}$. Then, we get that $x \in \mathcal{X}_{s'}$ and $d \in \mathcal{D}_{s'}$ since there is only a single sort s' in Σ'. By definition of the downward closure, there exists some $(x, c) \in h \cap (\mathcal{X}_{s'} \times \mathcal{D}_{s'})$ such that $d \preceq_{s'} c$. From Lemma 1 it follows that $\preceq_{s'}$ is reflexive, and thus $(x, c) \in h$ and $c \preceq_{s'} c$ imply $(x, c) \in h{\downarrow_\Sigma}$. Since $\langle h, t\rangle$ is an interpretation over Σ, we get that $(x, c) \in t{\downarrow_\Sigma}$. By Lemma 5, it follows that $(x, c) \in t{\downarrow_{\Sigma'}}$. Then, by definition of the downward closure, it follows that there exists some $(x, c') \in t \cap (\mathcal{X}_{s'} \times \mathcal{D}_{s'})$ such that $c \preceq_{s'} c'$ and by Lemma 1 we get that $\preceq_{s'}$ is transitive which implies $d \preceq_{s'} c'$. Therefore, we get that $(x, d) \in t{\downarrow_{\Sigma'}}$, and thus $\langle h, t\rangle$ is an interpretation over Σ'.

For the *right-to-left* direction, assume that $\langle h, t\rangle$ is an interpretation over Σ' which satisfies $\langle h, t\rangle \models \bigcup_{s\in\mathcal{S}} \{x : \mathcal{D}_{s'} \to x : \mathcal{D}_s \mid x \in \mathcal{X}_s\}$. From $\langle h, t\rangle \models \bigcup_{s\in\mathcal{S}} \{x : \mathcal{D}_{s'} \to x : \mathcal{D}_s \mid x \in \mathcal{X}_s\}$ it follows that $(x, c) \in v$ implies $(x, c) \in \mathcal{X}_s \times \mathcal{D}_s$ for $v \in \{h, t\}$ and for some $s \in \mathcal{S}$. Therefore, h and t are valuations over Σ.

It is left to show that $h{\downarrow_\Sigma} \subseteq t{\downarrow_\Sigma}$. Pick any $(x, d) \in h{\downarrow_\Sigma}$. Then, we get that $x \in \mathcal{X}_s$ and $d \in \mathcal{D}_s$ for some $s \in \mathcal{S}$. By Lemma 5, it follows that $(x, d) \in h{\downarrow_{\Sigma'}}$, and since $\langle h, t\rangle$ is an interpretation over Σ' it follows that $(x, d) \in t{\downarrow_{\Sigma'}}$. By definition of the downward closure, it follows that there exists some $(x, c) \in t \cap (\mathcal{X}_{s'} \times \mathcal{D}_{s'})$ such that $d \preceq_{s'} c$ Since t is a valuation over Σ and $(x, c) \in t$, it follows that $c \in \mathcal{D}_s$. We need to show that $d \preceq_s c$.

From the construction of Σ', it follows that there exists some sort $s'' \in \mathcal{S}$ such that $d \preceq_{s''} c$. We proceed by cases.

Case 1. $s'' = s$. Then, $d \preceq_s c$ follows directly.

Case 2. $s'' \neq s$. Then, we get that $c, d \in \mathcal{D}_{s''}$. Since s and s'' are compatible, we get that $\preceq_s|_{\mathcal{D}_s \cap \mathcal{D}_{s''}} = \preceq_{s''}|_{\mathcal{D}_s \cap \mathcal{D}_{s''}}$. Therefore, from $c, d \in \mathcal{D}_s \cap \mathcal{D}_{s''}$ and $d \preceq_{s''} c$ it follows that $d \preceq_s c$.

In both cases, we get that $d \preceq_s c$ which implies $(x, d) \in t{\downarrow_\Sigma}$, and thus $\langle h, t\rangle$ is an interpretation over Σ.

Proof of Proposition 2 For the *left-to-right* direction, let t be a Σ-stable model of Γ. Then $\langle t, t\rangle \models \Gamma$ and there is no valuation h over Σ such that $h{\downarrow_\Sigma} \subset t{\downarrow_\Sigma}$ and

$\langle h,t\rangle \models \Gamma$. By Lemma 6, we get that $\langle t,t\rangle \models \bigcup_{s\in\mathcal{S}}\{x:\mathcal{D}_{s'} \to x:\mathcal{D}_s \mid x\in\mathcal{X}_s\}$, and thus $\langle t,t\rangle \models \Gamma'$.

Next, we show that for every $h{\downarrow_{\Sigma'}} \subset t{\downarrow_{\Sigma'}}$, we get that $\langle h,t\rangle \not\models \Gamma'$. Pick any interpretation $\langle h,t\rangle$ over Σ' such that $h{\downarrow_{\Sigma'}} \subset t{\downarrow_{\Sigma'}}$ and assume that $\langle h,t\rangle \models \Gamma'$. By Lemma 2, we get that $h \neq t$. We show that this leads to a contradiction. From $\langle h,t\rangle \models \Gamma'$ we get that $\langle h,t\rangle \models \bigcup_{s\in\mathcal{S}}\{x:\mathcal{D}_{s'} \to x:\mathcal{D}_s \mid x\in\mathcal{X}_s\}$ and by Lemma 6 it follows that $\langle h,t\rangle$ is an interpretation over Σ. Thus, we get that $h{\downarrow_{\Sigma}} \subseteq t{\downarrow_{\Sigma}}$, and from $h\neq t$ and by Lemma 2, we get that $h{\downarrow_{\Sigma}} \subset t{\downarrow_{\Sigma}}$. Then, from our assumption it follows that $\langle h,t\rangle \models \Gamma$ which is a contradiction with $\langle t,t\rangle$ being a Σ-stable model of Γ. Therefore, $\langle h,t\rangle \not\models \Gamma'$. Since $\langle h,t\rangle$ was picked arbitrarily, it follows that t is a Σ-stable model of Γ.

For the *right-to-left* direction, let t be an Σ'-stable model of Γ'. Then, $\langle t,t\rangle \models \Gamma'$ and there is no valuation h over Σ such that $h{\downarrow_{\Sigma'}} \subset t{\downarrow_{\Sigma'}}$ and $\langle h,t\rangle \models \Gamma'$. From $\langle t,t\rangle \models \bigcup_{s\in\mathcal{S}}\{x:\mathcal{D}_{s'} \to x:\mathcal{D}_s \mid x\in\mathcal{X}_s\}$ and Lemma 6 it follows that $\langle t,t\rangle$ is an interpretation over Σ. We show that t is a Σ-stable model of Γ.

First, from $\langle t,t\rangle \models \Gamma'$ it follows that $\langle t,t\rangle \models \Gamma$. Let $\langle h,t\rangle$ be an interpretation over Σ such that $h{\downarrow_{\Sigma}} \subset t{\downarrow_{\Sigma}}$. By Lemma 2, we get that $h\neq t$. From Lemma 6 it follows that $\langle h,t\rangle$ is an interpretation over Σ' which satisfies $\langle h,t\rangle \models \bigcup_{s\in\mathcal{S}}\{x:\mathcal{D}_{s'} \to x:\mathcal{D}_s \mid x\in\mathcal{X}_s\}$. Therefore, we get that $h{\downarrow_{\Sigma'}} \subseteq t{\downarrow_{\Sigma'}}$, and from $h\neq t$ and by Lemma 2 it follows that $h{\downarrow_{\Sigma'}} \subset t{\downarrow_{\Sigma'}}$. Then, from our assumption that t is a Σ'-stable model of Γ' we get that $\langle h,t\rangle \not\models \Gamma'$. Since $\langle h,t\rangle \models \bigcup_{s\in\mathcal{S}}\{x:\mathcal{D}_{s'} \to x:\mathcal{D}_s \mid x\in\mathcal{X}_s\}$ it has to hold that $\langle h,t\rangle \not\models \Gamma$. Lastly, as $\langle h,t\rangle$ was picked arbitrarily, it follows that t is a Σ-stable model of Γ. □

A.3 Proof of Theorem 1

We start by reviewing the definition of HT$_c$ as given in [5]. In HT$_c$, a CSP is expressed as a triple $\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle$, also called *signature*, where $\mathcal{X}$ is a set of *variables* over some non-empty *domain* $\mathcal{D}$, and $\mathcal{C}$ is a set of *constraint atoms*. We extend the domain by the special symbol $\boldsymbol{u} \notin \mathcal{D}$, denoting undefinedness, through $\mathcal{D}_{\boldsymbol{u}} \stackrel{\text{def}}{=} \mathcal{D}\cup\{\boldsymbol{u}\}$. An HT$_c$-valuation v over $\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle$ is a function $v:\mathcal{X}\to\mathcal{D}_{\boldsymbol{u}}$. We define the set of all HT$_c$-valuations over $\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle$ as $\mathcal{V}^{\mathcal{X},\mathcal{D}}$, or simply as $\mathcal{V}$ when clear from context. As for HT$_b$ we can view a valuation v as a set of pairs $(x,v(x))\in\mathcal{X}\times\mathcal{D}$ which drops any pairs $(x,\boldsymbol{u})$.

The semantics of constraint atoms in HT$_c$ is defined via denotations, which are functions of form $[\![\cdot]\!]_{\mathcal{X},\mathcal{D}}:\mathcal{C}\to 2^{\mathcal{V}}$. A formula φ over signature $\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle$ is defined as

$$\varphi ::= \bot \mid c \mid \varphi\wedge\varphi \mid \varphi\vee\varphi \mid \varphi\to\varphi \quad \text{where } c\in\mathcal{C}\,,$$

with $\top$ and $\neg\varphi$ defined as in Section 2. A theory in HT$_c$ is a set of formulas over $\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle$. An HT$_c$-interpretation over $\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle$ is a pair $\langle h,t\rangle$ of valuations over $\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle$ such that $h\subseteq t$.

We define satisfaction in HT$_c$ as follows:

Definition 3. *Let $\langle h,t\rangle$ be an HT$_c$-interpretation over $\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle$ and φ be a formula over $\mathcal{C}$.*

Then, $\langle h,t\rangle$ satisfies φ, written $\langle h,t\rangle \models_c \varphi$, if the following holds:

1. $\langle h,t\rangle \not\models_c \bot$
2. $\langle h,t\rangle \models_c c$ *iff* $v \in [\![c]\!]_{\mathcal{X},\mathcal{D}}$ *for atom* $c \in \mathcal{C}$ *and for all* $v \in \{h,t\}$
3. $\langle h,t\rangle \models_c \varphi_1 \wedge \varphi_2$ *iff* $\langle h,t\rangle \models_c \varphi_1$ *and* $\langle h,t\rangle \models_c \varphi_2$
4. $\langle h,t\rangle \models_c \varphi_1 \vee \varphi_2$ *iff* $\langle h,t\rangle \models_c \varphi_1$ *or* $\langle h,t\rangle \models_c \varphi_2$
5. $\langle h,t\rangle \models_c \varphi_1 \rightarrow \varphi_2$ *iff* $\langle v,t\rangle \not\models_c \varphi_1$ *or* $\langle v,t\rangle \models \varphi_2$ *for both* $v \in \{h,t\}$

This definition differs from the one in [5] in Condition 2 where we only check in h for satisfaction of a constraint atom.

2. $\langle h,t\rangle \models_c c$ iff $h \in [\![c]\!]_{\mathcal{X},\mathcal{D}}$ for atom $c \in \mathcal{C}$

Note that [5] imposes the following restriction on denotations: For all $c \in \mathcal{C}$, $x \in \mathcal{X}$, and $v, v' \in \mathcal{V}^{\mathcal{X},\mathcal{D}}$ $v \in [\![c]\!]_{\mathcal{X},\mathcal{D}}$ the following condition holds:

1. $v \in [\![c]\!]_{\mathcal{X},\mathcal{D}}$ and $v \subseteq v'$ imply $v' \in [\![c]\!]_{\mathcal{X},\mathcal{D}}$.

Under this restriction, Definition 3 is equivalent to the definition of the satisfaction relation in [5].

For the remainder of the paper, we use the more general satisfaction relation from Definition 3 and do not impose any conditions on denotations. An interpretation $\langle h,t\rangle$ is a model of a theory Γ in HT$_c$, written $\langle h,t\rangle \models_c \Gamma$, when $\langle h,t\rangle \models_c \varphi$ for all $\varphi \in \Gamma$.

Definition 4. *An HT$_c$-interpretation $\langle t,t\rangle$ over $\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle$ is a equilibrium model of a theory Γ in iff $\langle t,t\rangle \models_c \Gamma$ and there is no $h \subset t$ such that $\langle h,t\rangle \models_c \Gamma$.*

As above, we call t a stable model of Γ in HT$_c$, if $\langle t,t\rangle$ is an equilibrium model of Γ.

We prove that any HT$_c$-theory can be expressed as a single-sorted HT$_b$-theory such that the equilibrium models coincide. For any HT$_c$-signature $\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle$, we construct a single-sorted HT$_b$-signature $\Sigma_{\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle} = \langle \{s\}, \mathcal{X}_s, (\mathcal{D}_s, id_s), \mathcal{C}\rangle$, with $\mathcal{X}_s = \mathcal{X}$ and $\mathcal{D}_s = \mathcal{D}$, and $id_s = \{(d,d) \mid d \in \mathcal{D}_s\}$. It is easy to see that any formula in HT$_c$ over $\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle$ is also a formula in HT$_b$ over $\Sigma_{\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle}$, and vice versa. Note that every HT$_c$-valuation v over $\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle$ is an HT$_b$-valuation over $\Sigma_{\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle}$ and vice versa,

Lemma 7. *Let v and v' be two HT$_c$-valuations over $\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle$ (alternatively, HT$_b$-valuations over $\Sigma_{\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle}$). Then, $v \subseteq v'$ iff $v{\downarrow} \subseteq v'{\downarrow}$.*

Proof. Follows directly from Lemma 3. □

Now, we can formulate the following result.

Proposition 4. *Let $\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle$ be an HT$_c$-signature and let Γ be a theory over $\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle$. For every $c \in \mathcal{C}$, assume that $[\![c]\!]_{\mathcal{X},\mathcal{D}} = [\![c]\!]_{\Sigma_{\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle}}$.*

Then, the following holds:

1. $\langle h,t\rangle$ *is an interpretation over* $\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle$ *in* HT_c *iff* $\langle h,t\rangle$ *is an interpretation over signature* $\Sigma_{\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle}$ *in* HT_b *and*
2. $\langle h,t\rangle \models_c \Gamma$ *iff* $\langle h,t\rangle \models \Gamma$.

Proof. First note that $[\![\, c \,]\!]_{\mathcal{X},\mathcal{D}} = [\![\, c \,]\!]_{\Sigma_{\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle}}$ for every $c \in \mathcal{C}$ is a valid assumption because both signatures use the same set of variables and domain elements, and we do not impose any conditions on denotations.

The first statement follows directly from Lemma 7.

For the second statement, note that any HT_c-theory is also an HT_b-theory. It is sufficient to prove the base case as the other cases follow by induction over the structure of formulas. Let $c \in \mathcal{C}$ be a constraint atom. Then,

$$
\begin{aligned}
&\langle h,t\rangle \models_c c \\
&\text{iff } h \in [\![\, c \,]\!]_{\mathcal{X},\mathcal{D}} \text{ and } t \in [\![\, c \,]\!]_{\mathcal{X},\mathcal{D}} && \text{(Definition 3)}\\
&\text{iff } h \in [\![\, c \,]\!]_{\Sigma_{\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle}} \text{ and } t \in [\![\, c \,]\!]_{\Sigma_{\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle}} && ([\![\, c \,]\!]_{\mathcal{X},\mathcal{D}} = [\![\, c \,]\!]_{\Sigma_{\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle}})\\
&\text{iff } \langle h,t\rangle \models c && \text{(Definition 1)}\,.
\end{aligned}
$$

□

Proof of Theorem 1 Let Γ be an HT_c-theory over $\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle$.

For the *left-to-right* direction, assume that t is a stable model of Γ in HT_c. Then, by definition, we get that $\langle t,t\rangle \models_c \Gamma$ and for any $h \subset t$ it holds that $\langle h,t\rangle \not\models_c \Gamma$. By Proposition 4.1 it follows that $\langle t,t\rangle$ is an HT_b-interpretation over $\Sigma_{\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle}$ and by Proposition 4.2, we get that $\langle t,t\rangle \models \Gamma$. Next, let $\langle h,t\rangle$ be an HT_b-interpretation over $\Sigma_{\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle}$ and assume that $h{\downarrow} \subset t{\downarrow}$. We show that $\langle h,t\rangle \not\models \Gamma$.

By Proposition 4.1, we get that $\langle h,t\rangle$ is an HT_c-interpretation over $\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle$. Furthermore, by Lemma 3 it follows that $t{\downarrow}= t$ and $h{\downarrow}= h$, and thus $h{\downarrow} \subset t{\downarrow}$ implies $h \subset t$. Since t is a stable model of Γ in HT_c, it follows that $\langle h,t\rangle \not\models_c \Gamma$. Then, by Proposition 4.2 we get that $\langle h,t\rangle \not\models \Gamma$, and thus t is a $\Sigma_{\langle \mathcal{X},\mathcal{D},\mathcal{C}\rangle}$-stable model of Γ in HT_b. The other direction follows by the same arguments. □